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Integrability and fusion algebra for quantum mappings

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Abstract. We apply the fusion procedure to a quantum Yang–Baxter algebra associated with time-discrete integrable systems, notably integrable quantum mappings. We present a general construction of higher-order quantum invariants for these systems. As an important class of examples, we present the Yang–Baxter structure of the Gel'fand–Dikii mapping hierarchy that we have introduced in previous papers, together with the corresponding explicit commuting family of quantum invariants.

1. Introduction

Discrete integrable quantum models, in which the spatial variable is discretized, have played an important role in the development of the quantum inverse scattering method [1, 2], and subsequently in the advent of quantum groups [3–7]. In the modern developments in string theory and conformal field theory such models also play a particularly important role [8–10]. However, quantum models, in which also the time-flow is discretized (i.e. whose classical counterparts are integrable partial difference equations), have not been studied widely until recently [11]. The study of integrable systems with a discrete time evolution is certainly not new. Exactly solvable lattice models in statistical mechanics form a widely studied class of such systems, the transfer matrix being the operator that generates the discrete-time flow. Integrable partial difference equations (discrete counterparts of soliton systems) have also been studied, ([12] and references therein). Recently, these systems have become of interest in connection with the construction of exactly integrable *mappings*, i.e. finite-dimensional systems with discrete time-flow [13, 14]. Their integrability is to be understood in the sense that the discrete time-flow is the iterate of a canonical transformation, preserving a suitable symplectic form, and which carries exact invariants which are in involution with respect to this symplectic form. Other types of integrable mappings have been considered also in the recent literature [15] for a review).

Recently, a theory of integrable *quantum mappings* was formulated in [16, 17] (cf also [18]). These are the discrete-time quantum systems which are obtained from the classical mappings by quantization within the Yang–Baxter formalism. For the discrete-time systems it turns out that a beautiful structure arises: it was pointed out in

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[17], that in contrast to the continuous-time quantum system, in the case of mappings one has to take into account the spatial as well as temporal part of the quantum Lax system that governs the time-evolution. However, the full quantum Yang–Baxter structure containing both parts carries a consistent set of universal algebraic relations, reminiscent of the algebraic structure that was proposed for the cotangent bundle of a quantum group [19].

For the mappings of $k\alpha v$ and $m\kappa\alpha v$ type that were considered in [16] and [18], a construction of exact quantum invariants is given by expanding a quantum deformation of the trace of the monodromy matrix of the model. This will yield in principle a large enough family of commuting quantum invariants to ‘diagonalize’ the discrete-time model. The deformation, introduced by a quantum K -matrix (in the spirit of [20], cf also [21]), is established quite straightforwardly from the relations supplied by the full Yang–Baxter structure. However, in most examples of quantum mappings, notably those related to the higher-order members of the Gel’fand–Dikiĭ (GD) hierarchy [12, 17], one may expect that the trace of the monodromy matrix does not yield a large enough family of commuting invariants. In those cases, one needs the analogues (i.e. the K -deformed versions) of the higher-order quantum minors and quantum determinants, to supply the remaining invariants. It is the purpose of this paper to give a general construction of higher-order quantum invariants of integrable mappings within the mentioned full Yang–Baxter structure.

The outline of this paper is as follows. In section 2, we briefly summarize the full (spatial and temporal) Yang–Baxter structure for mappings. In section 3, we develop the fusion algebra, i.e. the algebra of higher-order tensor products of generators of the extended Yang–Baxter algebra. In section 4, we use the fusion relations to give a construction of exact higher-order quantum invariants, corresponding to the K -deformed quantum minors and determinants. Furthermore, we give a proof that these will yield a commuting family of quantum operators. Finally, in section 5, we apply the general construction to obtain the quantum invariants of the mappings in the Gel’fand–Dikiĭ hierarchy. In order to make the paper self-contained, we supply proofs of most of the basic relations, even though we have no doubt that some of these must be known to the experts.

2. Integrable quantum mappings

We focus on the quantization of the mappings that arise from integrable partial difference equations by finite-dimensional reductions. Both on the classical as well as on the quantum level they arise as the compatibility conditions of a discrete-time zS (Zakharov–Shabat) system

$$L'_n(\lambda)M_n(\lambda) = M_{n+1}(\lambda)L_n(\lambda) \quad (2.1)$$

in which λ is a spectral parameter, L_n is the lattice translation operator at site n , and the prime denotes the discrete time-shift corresponding to another translation on a two-dimensional lattice. As L and M , in the quantum case, depend on quantum operators (acting on some well-chosen Hilbert space \mathcal{H}), the question of operator ordering becomes important. Throughout we impose as a normal order the order which is induced by the lattice enumeration, with n increasing from the left to the right. Finite-dimensional mappings are obtained from (2.1) imposing a periodicity condition $L_n(\lambda) = L_{n+P}(\lambda)$, $M_n(\lambda) = M_{n+P}(\lambda)$, for some period P .

We summarize here briefly the (non-ultralocal) Yang–Baxter structure for integrable quantum mappings, that was introduced in [17].

Yang–Baxter structure. The quantum L -operator for the mappings that we consider here obeys commutation relations of non-ultralocal type, i.e.

$$R_{12}^+ L_{n,1} L_{n,2} = L_{n,2} L_{n,1} R_{12}^- \tag{2.2a}$$

$$L_{n+1,1} S_{12}^+ L_{n,2} = L_{n,2} L_{n+1,1} \tag{2.2b}$$

$$L_{n,1} L_{m,2} = L_{m,2} L_{n,1}, \quad |n - m| \geq 2 \tag{2.2c}$$

with non-trivial commutation relations only between L -operators on the same site and on neighbouring sites.

The notation we use is the following. R_{ij}^\pm and S_{ij}^\pm are matrices in $\text{End}(V_i \otimes V_j) \rightarrow \text{End} \otimes_a V_a$, where V_i and V_j are vector spaces, typically the representation spaces of some quantum group, but here to be concrete we take them to be different copies of $V = \mathbb{C}^N$, and both R^\pm and S^\pm in $\text{End}(V \otimes V)$, the indices referring to the embedding $\text{End}(V \otimes V) \hookrightarrow \text{End}(\otimes_a V_a)$, in a tensor product of a yet unspecified number of factors. We adopt the usual convention that the subscripts 1, 2, ... denote factors in a matricial tensor product, i.e. $A_{i_1, i_2, \dots, i_M} = A_{i_1, i_2, \dots, i_M}(\lambda_1, \lambda_2, \dots, \lambda_M)$ denotes a matrix acting non-trivially only on the factors labelled by i_1, i_2, \dots, i_M of a tensor product $\otimes_a V_a$, of vector spaces V_a and trivially on the other factors. Whenever there is no cause for ambiguity, we suppress in the notation the explicit dependence on the spectral parameters λ_i , adopting the convention that each one accompanies its respective factor in the tensor product. Thus, for instance, $L_{n,1}$ and $L_{n,2}$ are shorthand notations for $L_n(\lambda_1) \otimes 1$, respectively, $1 \otimes L_n(\lambda_2)$. In section 5, we will specify the R and S matrix even further, adapting us to the special example of systems associated with the \mathfrak{GD} hierarchy. The quantum L matrix is taken in $\text{End}(V \otimes \mathfrak{h})$, where V acts as the auxiliary space (in the terminology of [22]), and \mathfrak{h} is the Hilbert space of the quantum system.

The quantum relations (2.2) were introduced for the first time in [23] (cf also [24]) in connection with the quantum Toda theory, and in [25] for the quantum Wess–Zumino–Novikov–Witten (wznw) model. They define what in [26] is referred to as lattice current algebra (LCA), or quantum Kac–Moody algebra on the lattice. In [16], we introduced them in connection with the quantization of discrete-time models, namely to quantize mappings of KdV type.

The compatibility relations of the equations (2.2a, 2.2b) lead to the following set of consistency conditions for R^\pm and S^\pm

$$R_{12}^\pm R_{13}^\pm R_{23}^\pm = R_{23}^\pm R_{13}^\pm R_{12}^\pm \tag{2.3a}$$

$$R_{23}^\pm S_{12}^\pm S_{13}^\pm = S_{13}^\pm S_{12}^\pm R_{23}^\pm \tag{2.3b}$$

(i.e. two equations for the + sign, and two equations for the – sign), where $S_{12}^+ = S_{21}^-$. Equation (2.3a) is the quantum Yang–Baxter equation for R^\pm coupled with an additional equation (2.3b) for S^\pm . In addition to these relations we need to impose also

$$R_{12}^- S_{12}^- = S_{12}^+ R_{12}^+ \tag{2.4}$$

in order to be able to derive suitable commutation relations for the monodromy matrix of the systems under consideration in this paper.

As we are interested in the canonical structure of discrete-time integrable systems,

i.e. systems for which the time evolution is given by an iteration of a mapping, we need, as explained in [17], in addition to (2.2), commutation relations involving the discrete-time part of the zs system, namely the matrices M . These matrices containing quantum operators, we have a set of non-trivial commutation relations with the L -matrices which are such that the Yang-Baxter structure is preserved. Such a proposal, formulated in [17], consists of the following commutation relations in addition to the relations (2.2)

$$M_{n+1,1}S_{12}^+L_{n,2} = L_{n,2}M_{n+1,1} \quad (2.5a)$$

$$L'_{n,2}S_{12}^-M_{n,1} = M_{n,1}L'_{n,2} \quad (2.5b)$$

and

$$R_{12}^+M_{n,1}M_{n,2} = M_{n,2}M_{n,1}R_{12}^- \quad (2.6a)$$

$$M'_{n,1}S_{12}^+M_{n,2} = M_{n,2}M'_{n,1}. \quad (2.6b)$$

Some trivial commutation relations need also to be specified, namely

$$[M_{n,1}, L_{m,2}] = [M_{n+1,1}, L'_{m,2}] = [M_{n,1}, M_{m,2}] = [M'_{n+1,1}, M_{m,2}] = [M'_{n+1,1}, L_{m,2}] = 0 \quad (2.7)$$

($|n-m| \geq 2$), where the brackets denote matrix commutation. We shall not specify any other commutation relations: they do not belong to the Yang-Baxter structure, even though they might be non-trivial, as they depend on the details of specific models. The relations that we have given are self-contained in the sense that they are sufficient to show that the commutation relations (2.2) are preserved under the mapping coming from (2.1). Thus the quantum mappings associated with (2.1) are canonical in the sense that they leave the underlying Yang-Baxter structure invariant. Furthermore, it is shown in [27, 28] that for specific examples of quantum mappings that fit into the structure presented here (such as mappings associated with the lattice KdV and $MKdV$ equations [13]), there is a unitary operator that generates the mapping, acting on the quantum phase space of the system.

Quantum traces. To obtain quantum invariants of the mappings associated with the zs system (2.1), we need to introduce the monodromy matrix

$$T(\lambda) = \prod_{n=1}^P L_n(\lambda). \quad (2.8)$$

The commutation relations for the monodromy matrix are obtained from the relations for the L -matrix, (2.2), making use of the crucial relation (2.4), and taking into account the periodic boundary conditions. Thus we obtain

$$R_{12}^+T_1S_{12}^+T_2 = T_2S_{12}^-T_1R_{12}^-. \quad (2.9)$$

Equation (2.9) is similar to the commutation relations for the so-called algebra of currents of a quantum group [7, 29]. Versions of such algebras have been considered in different contexts, e.g. in connection with boundary conditions of integrable quantum chains [20, 21, 30, 31].

Following a treatment similar as the one in [20] (cf [18]), a commuting parameter-family of operators is obtained by taking

$$\tau(\lambda) = \text{tr}(T(\lambda)K(\lambda)) \quad (2.10)$$

for any family of numerical matrices $K(\lambda)$ obeying the relations

$$K_1^{\prime 1}(({}^1S_{12}^-)^{-1})K_2R_{12}^+ = R_{12}^-K_2^{\prime 2}(({}^2S_{12}^+)^{-1})K_1. \quad (2.11)$$

(We assume throughout that S_{12}^{\pm} and R_{12}^{\pm} are invertible). The left superscripts 1 and 2 denote the matrix transpositions with respect to the corresponding factors 1 and 2 in the matricial tensor product. Expanding (2.10) in powers of the spectral parameter λ , we obtain a set of commuting observables of the quantum system in terms of which we can find a common basis of eigenvectors in the associated Hilbert space.

The quantum mapping for the monodromy matrix, is simply given by the conjugation

$$T' = MTM^{-1} \tag{2.12}$$

where M is the M_n -matrix at the begin- and end-point of the chain, i.e. $M \equiv M_1 = M_{P+1}$. In general, M does not commute with the entries of T , as in our models M will non-trivially depend on quantum operators. Thus, the classical invariants obtained by taking the trace of the monodromy matrix, are no longer invariant on the quantum level. Therefore, we need to investigate the commutation relations between the monodromy matrix T and the matrix M . These are given by the equation

$$(TM^{-1})_1 S_{12}^+ M_2 = M_2 S_{12}^- (TM^{-1})_1 \tag{2.13}$$

in combination with

$$R_{12}^+ M_1 M_2 = M_2 M_1 R_{12}^- \tag{2.14}$$

where, by abuse of notation, M_2 denotes here not the matrix M at site $n=2$, but the matrix M_1 in the second factor of the tensorial product. (We use the same symbol for both M -matrices. It will be clear from the context which of the M -matrices we mean if we use them below.) It is interesting to note that the set of relations consisting of (2.9) and (2.13) is reminiscent of the relations describing the cotangent bundle of a quantum group $(T^*G)_q$ [19]. However, in the context of the present paper, we will refer to the algebra \mathcal{A} generated by T and M with defining relations (2.9), (2.13) and (2.14), simply as the quantum mapping algebra. As a consequence of (2.13), (2.14), the commutation relation (2.9) is preserved under the mapping (2.12). Furthermore, equation (2.13) can be used to find special solutions for K leading to exact quantum invariants. Introducing the permutation operator P_{12} acting in the tensor product of matrices $X(\lambda), Y(\lambda)$, by

$$P_{12} X_1 Y_2 = X_2 Y_1 P_{12} \quad (\lambda_1 = \lambda_2) \tag{2.15a}$$

interchanging the factors in the matricial tensor product in $\mathbb{C}^N \otimes \mathbb{C}^N$. For this operator we have the trace property

$$\text{tr}_1 P_{12} = 1_2, \tag{2.15b}$$

where 1_2 denotes the identity acting operator on vector space V_2 . In fact, introducing a tensor $K_{12} = P_{12} K_1 K_2$, choosing $\lambda_1 = \lambda_2$, we can take the trace over both factors in the tensor product, contracting both sides of (2.13) with K_{12} . This leads to an exact quantum invariant of the form (2.10), provided that the matrix $K(\lambda)$ solves the condition

$$\text{tr}_1 (P_{12} K_2 S_{12}^+) = 1_2 \tag{2.16}$$

in which tr_1 denotes the trace over the first factor of the tensor product. Equation (2.16) is explicitly solved by taking

$$K_2 = \text{tr}_1 \{ P_{12}^{-1} ((^1 S_{12}^+)^{-1}) \} \tag{2.17}$$

and this solution $K(\lambda)$ can be shown to obey also (2.11). Hence, the invariants obtained from (2.10) by expanding in powers of the spectral parameter λ form a commuting family of operators. For the quantum mappings considered in [16, 18], notably mappings associated with the quantum lattice $\kappa\alpha V$ and modified $\kappa\alpha V$ systems, we obtain in this way enough invariants to establish integrability. However, for the more general class of $N \times N$ models presented in [17], we need additional invariants. For this we must develop the fusion algebra associated with the quantum algebra given by equations (2.9), (2.13) and (2.14).

3. Fusion procedure

The structure of the mappings outlined in the previous section holds for a large class of systems, notably the mappings associated with the lattice Gel'fand–Dikii hierarchy [12]. However, for the higher members of this hierarchy it is not sufficient to consider only the trace (2.10) to generate exact quantum invariants. In fact, for these systems one needs also higher-order invariants corresponding roughly to the trace of powers of the monodromy matrix. The construction of such higher-order commuting families of operators corresponding to the quantum analogue of the classical object $\text{tr}(T^n)$, i.e. traces over powers of the monodromy matrix, is called fusion procedure [32], and in particular leads to the proper definition of quantum determinants and minors [33] (cf also [6, 7, 20, 34]). Some of the results in this section were also obtained in [35] in the special case of twisted Yangians (cf also [36]). Fusion is also used to obtain tensor products of representations of quantum algebras, notably of affine quantum groups. The latter connection, however, is not of direct concern to us here.

Notation. The objects we need to build for the fusion algebra are tensorial products of matrices, i.e. they are objects of the form $A_a = A_{(i_1, i_2, \dots, i_n)}$ and $A_{a,b} = A_{(i_1, \dots, i_n, j_1, \dots, j_m)}$, depending on a multiple indices $a = (i_1, i_2, \dots, i_n)$ and $b = (j_1, \dots, j_m)$, labelling the factors in a tensor product of vector-spaces $\otimes_a V_a$, on which A_a and $A_{a,b}$ act non-trivially. We can think of these vector spaces as being irreducible modules of the quantum algebra introduced in the previous section. However, in order to be concrete, we shall take them to be different copies of $V = \mathbb{C}^N$, corresponding to the fundamental representation of specific realisations of the algebra, that we will consider below.

We can now introduce the following formal scheme of tensorial objects labelled by multi-indices. First we distinguish between an elementary index, denoted by i_1, i_2, \dots , and multi-indices made up from elementary indices. The elementary indices correspond to the labels of single vector spaces, typically irreducible modules of the quantum algebra, whereas the multi-indices correspond to tensor products of these vector spaces. Next, we introduce some manipulations on multi-indices allowing us to build objects that are labelled by its entries. Thus, if a denotes such a multi-index, i.e. an ordered tuple $a = (i_1, i_2, \dots, i_n)$, then we denote by \hat{a} the multiple index corresponding to the reverse order of labels, i.e. $\hat{a} = (i_n, \dots, i_2, i_1)$. Let us denote by $\ell(a)$ the length of the multi-index a , i.e. $\ell(a) = n$ for $a = (i_1, i_2, \dots, i_n)$. Furthermore, we can join multi-indices as words in a free algebra on a set, namely if a and b are multi-indices $a = (i_1, i_2, \dots, i_n)$ and $b = (j_1, j_2, \dots, j_m)$, then we denote by (ab) the multi-index obtained by merging the two together, i.e. $(ab) = (i_1, i_2, \dots, i_n, j_1, j_2, \dots, j_m)$.

We can now build a hierarchy of tensorial objects, labelled multi-indices, starting from an object A_{ij} depending on two elementary indices. There are two types of objects that are of interest, namely *one*-multi-index objects A_a and *two*-multi-index objects $A_{a,b}$. They are generated from A_{ij} in a recursive way by following the rules

$$A_{(ab)} = A_b A_{a,b} A_a \tag{3.1}$$

and

$$A_{(ab),c} = A_{a,c} A_{b,c} \quad A_{a,(bc)} = A_{a,b} A_{a,c} \tag{3.2}$$

and adopting the convention that we take $A_i \equiv 1_i$, i.e. the objects A depending on a single elementary index act as the unit matrix on the corresponding vector space. (Note also that one has to distinguish the two-multi-index object $A_{a,b}$ from the one-multi-index object $A_{(ab)}$.) Thus for example, in building these tensorial objects, we will obtain

$$A_{i_1, \dots, i_n} = (A_{i_{n-1}, i_n})(A_{i_{n-2}, i_n} A_{i_{n-2}, i_{n-1}}) \dots (A_{i_1, i_n} A_{i_1, i_{n-1}} \dots A_{i_1, i_2})$$

and

$$A_{(i_1, \dots, i_n), (j_1, \dots, j_m)} = (A_{i_1, j_1} \dots A_{i_n, j_1}) \dots (A_{i_1, j_m} \dots A_{i_n, j_m}).$$

However, the recursive relations (3.1) and (3.2) will be more useful for us than these explicit expressions.

Fundamental relations. Let us take for A now objects like R_{12}^\pm , obeying the Yang–Baxter equation (2.3a) or S_{12}^\pm obeying (2.3b), as well as the important relation (2.4). Then, we can generate, using the prescriptions (3.1), multi-index objects R_a, S_a , depending on a single multi-index, or objects $R_{a,b}$ and $S_{a,b}$ depending on double multi-index according to (3.2). For these objects we can derive the Yang–Baxter type of relations

$$R_{a,b}^\pm R_{a,c}^\pm R_{b,c}^\pm = R_{b,c}^\pm R_{a,c}^\pm R_{a,b}^\pm \tag{3.3a}$$

$$R_{b,c}^\pm S_{a,b}^\pm S_{a,c}^\pm = S_{a,c}^\pm S_{a,b}^\pm R_{b,c}^\pm. \tag{3.3b}$$

In addition, we can derive the identities

$$R_a^\pm R_{a,b}^\pm = R_{a,b}^\pm R_a^\pm \tag{3.4a}$$

$$R_a^\pm R_{b,a}^\pm = R_{b,a}^\pm R_a^\pm \tag{3.4b}$$

$$R_a^\pm S_{b,a}^\pm = S_{b,a}^\pm R_a^\pm. \tag{3.4c}$$

In other words, R_a^\pm reverses the multi-index a when it is pulled through an $R_{a,b}^\pm$ or an $S_{a,b}^\pm$. Then, there are relations that can be derived starting from (2.4), namely

$$R_{a,b}^- S_{(ba)}^+ = S_{(ab)}^+ R_{a,b}^+ \tag{3.5}$$

as well as

$$S_a^+ R_a^+ = R_a^- S_a^+. \tag{3.6}$$

In other words, S_a^+ reverses the sign when it is pulled through an R_a^\pm or $R_{a,b}^\pm$. Equations (3.3)–(3.6), which are proven in appendix 1 provide us with the complete set of relations that we need to be able to define the fusion algebra for the quantum mappings.

Fusion algebra. Apart from the multi-index tensor objects that can be generated from

an elementary two-index object, like S_{12}^\pm or R_{12}^\pm , we have multi-index objects that are generated from a single-index object like T_1 or M_1 . These are iteratively defined by the relations

$$T_{(ab)} = R_{a,b}^+ T_a S_{a,b}^+ T_b \quad M_{(ab)} = M_a M_b. \tag{3.7}$$

From the relations (2.9), (2.6a), (2.13) for objects T and M , depending on an elementary index, one can derive by iteration the following set of relations

$$R_{a,b}^+ T_a S_{a,b}^+ T_b = T_b S_{a,b}^- T_a R_{a,b}^- \tag{3.8a}$$

$$R_{a,b}^+ M_a M_b = M_b M_a R_{a,b}^- \tag{3.8b}$$

$$[T_a M_a^{-1} (S_a^+)^{-1}] S_{a,b}^+ M_b = M_b S_{a,b}^- [T_a M_a^{-1} (S_a^+)^{-1}] \tag{3.8c}$$

$$R_a^+ M_a = M_a R_a^- \tag{3.8d}$$

Furthermore, as a consequence of (3.7), with the use of (3.4), we have that T_a is of the form

$$T_a = R_a^+ T_a^0 = T_a^0 R_a^- \quad T_{(ab)}^0 = T_a^0 S_{a,b}^+ T_b^0. \tag{3.9}$$

The quantum mapping obtained by iteration of (2.12), with the use of (3.8), for the multi-index monodromy matrix T_a , is given by

$$T_a^+ S_a^+ M_a = M_a S_a^+ T_a. \tag{3.10}$$

Equations (3.8), together with the definitions (3.7), and the mapping (3.10), define a complete set of relations by which we can describe higher-order tensor products of the algebra generated by M and T as given by the relations (2.9)–(2.14). We shall refer to equations (3.8) as the defining relations for the fusion algebra for the quantum mappings described by (2.12). After giving a proof of these equations, we will use them in order to generate higher-order commuting families of exact quantum invariants of the mappings defined by (3.10).

Proof of relations (3.8). To prove equations (3.8) it is most convenient to break down the relations for multi-index objects depending on joint indices (ab) into separate parts, assuming that the relations hold for these parts (by induction). Thus, to prove (3.8a) we can perform the following sequence of steps

$$\begin{aligned} & R_{(ab),c}^+ T_{(ab)} S_{(ab),c}^+ T_c \\ &= R_{b,c}^+ R_{a,c}^+ R_{a,b}^+ T_a S_{ab}^+ T_b S_{a,c}^+ S_{b,c}^+ T_c \\ &= R_{b,c}^+ R_{a,c}^+ T_b S_{a,b}^- T_a R_{a,b}^- S_{a,c}^+ S_{b,c}^+ T_c \\ &= R_{b,c}^+ R_{a,c}^+ T_b S_{a,b}^- T_a S_{b,c}^+ S_{a,c}^+ T_c R_{a,b}^- \\ &= R_{b,c}^+ T_b S_{b,c}^+ S_{a,b}^- R_{a,c}^+ T_a S_{a,c}^+ T_c R_{a,b}^- \\ &= R_{b,c}^+ T_b S_{b,c}^+ T_c S_{a,b}^- S_{a,c}^+ T_a R_{a,c}^- R_{a,b}^- \\ &= T_c S_{b,c}^- T_b S_{a,b}^- S_{a,c}^- R_{b,c}^- T_a R_{a,c}^- R_{a,b}^- \\ &= T_c S_{(ba),c}^- T_b S_{a,b}^- T_a R_{a,b}^- R_{a,c}^- R_{b,c}^- \\ &= T_c S_{(ab),c}^- T_{(ab)} R_{(ab),c}^- \end{aligned} \tag{3.11}$$

In a similar fashion one can show that

$$R_{a,(bc)}^+ T_a S_{a,(bc)}^+ T_{(bc)} = T_{(bc)} S_{a,(bc)}^- T_a R_{a,(bc)}^- \tag{3.12}$$

and (3.8a) follows from (3.11) and by (3.12) by induction. The consistency of the mapping (3.10) with the relations (3.8) is checked by the following calculation

$$\begin{aligned}
 T'_{(ab)} &= R_{a,b}^+ T_a S_{a,b}^+ T'_b \\
 &= R_{a,b}^+ M_a S_a^+ [T_a M_a^{-1} (S_a^+)^{-1} S_{a,b}^+ M_b] S_b^+ T_b M_b^{-1} (S_b^+)^{-1} \\
 &= R_{a,b}^+ M_a M_b S_a^+ S_{a,b}^- S_b^+ [T_a M_a^{-1} (S_a^+)^{-1}] [T_b M_b^{-1} (S_b^+)^{-1}] \\
 &= M_b M_a S_{(ab)}^+ R_{a,b}^+ T_a S_{b,a}^- [T_b M_b^{-1} (S_b^+)^{-1}] M_a^{-1} (S_{b,a}^+)^{-1} (S_a^+)^{-1} \\
 &= M_{(ab)}^+ S_{(ab)}^+ T_{(ab)} M_{(ab)}^{-1} (S_{(ab)}^+)^{-1}.
 \end{aligned} \tag{3.13}$$

Finally, to prove (3.8c), we note that

$$R_a^+ (TM^{-1})_a S_{a,b}^+ M_b = M_b S_{a,b}^- R_a^+ (TM^{-1})_a \tag{3.14}$$

in which we use the following notation for the iterate of the single-index matrix TM^{-1}

$$(TM^{-1})_{(ab)} = (TM^{-1})_a (TM^{-1})_b$$

together with the relation

$$R_a^+ (TM^{-1})_a = T_a M_a^{-1} (S_a^+)^{-1}. \tag{3.15}$$

This last relation can be proven inductively by the following steps

$$\begin{aligned}
 R_{(ab)}^+ (TM^{-1})_{(ab)} &= R_{a,b}^+ R_a^+ R_b^+ (TM^{-1})_a (TM^{-1})_b \\
 &= R_{a,b}^+ T_a M_a^{-1} (S_a^+)^{-1} T_b M_b^{-1} (S_b^+)^{-1} \\
 &= R_{a,b}^+ T_a S_{a,b}^- T_b M_b^{-1} M_a^{-1} (S_b^+)^{-1} (S_{b,a}^+)^{-1} (S_a^+)^{-1} = T_{(ab)} M_{(ab)}^{-1} (S_{(ab)}^+)^{-1}
 \end{aligned} \tag{3.16}$$

when we break up the indices into smaller parts. This concludes the induction argument in the proof of equations (3.8).

4. Quantum invariants

In order to make the formulas derived in the previous section more transparent we introduce for convenience some new objects, which seem to be slightly more natural. Thus, we define

$$\begin{aligned}
 \mathcal{T}_a &\equiv S_a^+ T_a & \mathcal{M}_a &\equiv S_a^+ M_a \\
 \mathcal{S}_{a,b}^+ &\equiv S_b^+ S_{a,b}^+ (S_b^+)^{-1} & \mathcal{S}_{a,b}^- &\equiv \mathcal{S}_{b,a}^+ \\
 \mathcal{R}_{a,b}^+ &\equiv S_a^+ S_b^+ R_{a,b}^+ (S_a^+)^{-1} (S_b^+)^{-1} & \mathcal{R}_{a,b}^- &\equiv \mathcal{R}_{a,b}^-.
 \end{aligned} \tag{4.1}$$

With these notations it is easily verified that (3.3) and (3.4) still hold, but now in terms of $\mathcal{R}_{a,b}^\pm$ and $\mathcal{S}_{a,b}^\pm$ instead of the original $R_{a,b}^\pm$ resp. $S_{a,b}^\pm$. However, (3.5) now reduces to

$$\mathcal{R}_{a,b}^- \mathcal{S}_{a,b}^- = \mathcal{S}_{a,b}^+ \mathcal{R}_{a,b}^+ \tag{4.2}$$

for which combination we have a ‘twisted’ Yang–Baxter relation

$$\mathcal{R}_{a,b}^- (\mathcal{S}_{a,c}^+ \mathcal{R}_{a,c}^+) \mathcal{S}_{a,b}^- (\mathcal{S}_{b,c}^+ \mathcal{R}_{b,c}^+) = (\mathcal{S}_{b,c}^+ \mathcal{R}_{b,c}^+) \mathcal{S}_{a,b}^+ (\mathcal{S}_{a,c}^+ \mathcal{R}_{a,c}^+) \mathcal{R}_{a,b}^+. \tag{4.3}$$

Furthermore, we have from (3.4) and (3.6)

$$\mathcal{R}_{a,b}^\pm R_a^- = R_a^- \mathcal{R}_{a,b}^\pm \tag{4.4a}$$

$$\mathcal{R}_{a,b}^\pm = R_b^- \mathcal{R}_{a,b}^\pm \tag{4.4b}$$

$$\mathcal{S}_{b,a}^\pm R_a^- = R_a^- \mathcal{S}_{b,a}^\pm. \tag{4.4c}$$

Using (4.1) we readily obtain from (3.8)

$$\mathcal{R}_{a,b}^+ \mathcal{T}_a \mathcal{P}_{a,b}^+ \mathcal{T}_b = \mathcal{T}_b \mathcal{P}_{a,b}^- \mathcal{T}_a \mathcal{R}_{a,b}^- \tag{4.5a}$$

$$\mathcal{R}_{a,b}^+ \mathcal{M}_a \mathcal{M}_b = \mathcal{M}_b \mathcal{M}_a \mathcal{R}_{a,b}^- \tag{4.5b}$$

$$\mathcal{T}_a \mathcal{M}_a^{-1} \mathcal{P}_{a,b}^+ \mathcal{M}_b = \mathcal{M}_b \mathcal{P}_{a,b}^- \mathcal{T}_a \mathcal{M}_a^{-1}. \tag{4.5c}$$

The mapping (3.1) adopts the more natural form

$$\mathcal{T}'_a \mathcal{M}_a = \mathcal{M}_a \mathcal{T}_a. \tag{4.6}$$

It is interesting to note the near resemblance between (4.5) and the original equations (2.9)–(2.14). Hence, in this form, the fusion algebra, defined by these relations, has the most convenient form to calculate quantum invariants following a prescription analogous to the construction of (2.10) and (2.17), together with (2.11). We also note the relation

$$\mathcal{T}_{(ab)} = S_{(ab)}^+ T_{(ab)} = (\mathcal{P}_{a,b}^+ \mathcal{R}_{a,b}^+) \mathcal{T}_a \mathcal{P}_{a,b}^+ \mathcal{T}_b. \tag{4.7}$$

Projectors. The relations (4.4) and (4.5) hold for arbitrary choice of the values of the spectral parameters $\lambda_a = (\lambda_1, \dots, \lambda_n)$, $\lambda_b = (\mu_1, \dots, \mu_m)$, associated with the factors denoted by the multi-indices $a = (i_1, \dots, i_n)$ and $b = (j_1, \dots, j_m)$ in the tensor products. In order to construct quantum invariants, we need now to impose certain relations between the values of the different $\lambda_1, \dots, \lambda_n$, and between the μ_1, \dots, μ_m . We now make the crucial assumption that for special choices of the spectral parameters $\lambda_a = (\lambda_1, \dots, \lambda_n)$ in R_a^\pm for $a = (i_1, \dots, i_n)$, the matrix R_a^- becomes a projector

$$(R_a^-)^2 = R_a^-. \tag{4.8}$$

This condition is satisfied in particular for models obeying the so-called ‘regularity condition’ [22]. For example, in the GD class of models that we consider in section 5, we have

$$R_{12}^-(\lambda, \lambda \pm h) = 1 \pm P_{12} \tag{4.9}$$

for some value $h \in \mathbb{C}$. In that case (cf also [33]) we obtain from $R_{(1, \dots, n)}^-$ a realization of the completely (anti) symmetric tensor

$$P_{(1, \dots, n)}^\pm = \frac{1}{n!} \sum_{\sigma \in S^n} (\pm 1)^\sigma P_\sigma \tag{4.10}$$

P_σ denoting the representation of the symmetric group S^n on the n -fold tensor product $(\mathbb{C}^N)^{\otimes n}$. In a more general context, we can think of these projectors as projecting out the various irreducible blocks in the tensor product of modules on the quantum algebra.

Furthermore, we note that from (3.9) we have

$$\mathcal{T}_a = R_a^- (S_a^+ T_a^0) = (S_a^+ T_a^0) R_a^- \quad R_a^- \mathcal{M}_a = \mathcal{M}_a R_a^- \tag{4.11}$$

using also (3.6). It is suggestive in the case of the special choice of parameters for which we have (4.8) to refer to the objects $\mathcal{T}_a = R_a^- \mathcal{T}_a R_a^-$ simply as *quantum minors*. In particular for situations in which we have (4.8), i.e. when R_a^- projects out an $(n + 1 - \ell(a))$ -dimensional subspace of the tensor product $V^{\otimes n}$ of the auxiliary vector space $V = \mathbb{C}^N$, the coefficients of \mathcal{T}_a become the actual quantum minors.

Construction of higher-order invariants. We now proceed by deriving the higher-order quantum invariants of the mapping, which is now encoded in (4.6). These are obtained by making direct use of (4.5c).

In the spirit of the above construction of multi-index objects, one can introduce multi-index permutation operators $\mathcal{P}_{a,b}$, generated from the elementary object P_{12} (the permutation matrix in the matricial tensor product $\mathbb{C}^N \otimes \mathbb{C}^N$) following the prescription in section 3. These operators $\mathcal{P}_{a,b}$ have the following properties when acting on arbitrary multi-index objects $X_a = X(\lambda_a)$

$$\begin{aligned} \text{tr}_a \mathcal{P}_{a,b} &= \text{tr}_a \mathcal{P}_{b,a} = 1_b & \mathcal{P}_{a,b} X_a &= X_b \mathcal{P}_{a,b} & \mathcal{P}_{a,b} X_b &= X_a \mathcal{P}_{a,b} \\ \mathcal{P}_{a,b} \mathcal{P}_{b,a} &= 1_{a,b} & (\ell(a) = \ell(b)) & & \lambda_a = \lambda_b \end{aligned} \tag{4.12}$$

in which $\lambda_a \equiv (\lambda_1, \dots, \lambda_n)$ denotes the collection of spectral parameters on which X_a depends, and denoting by tr_a the multiple trace over all factors in the tensor product labelled by the multi-index $a = (i_1, \dots, i_n)$.

We have now the following statement:

Proposition 1. If λ_a is chosen such that R_a^- is a projection matrix on the $\ell(a)$ -fold tensor product of vector spaces V_i , ($i = 1, \dots, n$), the family of operators

$$\tau^{(n)}(\lambda_a) \equiv \text{tr}_a(\mathcal{H}_a \mathcal{T}_a) \tag{4.13}$$

are exact invariants of the mapping (4.6), provided that \mathcal{H}_a solves the equation

$$\text{tr}_a(\mathcal{P}_{a,b} \mathcal{H}_b \mathcal{S}_{a,b}^+) = 1_b \tag{4.14}$$

where $\ell(a) = \ell(b) = n$, $\lambda_a = \lambda_b$.

Proof. Contracting both sides of (4.5c) with $\mathcal{P}_{a,b} \mathcal{H}_a \mathcal{L}_b$, in which \mathcal{H}_a and \mathcal{L}_b are (numerical) matricial tensors that need to be determined, and using (4.12), we find the equality

$$\begin{aligned} \text{tr}_{a,b} \{ \mathcal{P}_{a,b} \mathcal{H}_a \mathcal{L}_b \mathcal{T}_a \mathcal{M}_a^{-1} \mathcal{S}_{a,b}^+ \mathcal{M}_b \} &= \text{tr}_b \{ \mathcal{H}_b \mathcal{T}_b \mathcal{M}_b^{-1} \text{tr}_a(\mathcal{P}_{a,b} \mathcal{L}_b \mathcal{S}_{a,b}^+) \mathcal{M}_b \} \\ &= \text{tr}_{a,b} \{ \mathcal{P}_{a,b} \mathcal{H}_a \mathcal{L}_b \mathcal{M}_b \mathcal{S}_{a,b}^- \mathcal{M}_a^{-1} \mathcal{T}_a \} = \text{tr}_a \{ \mathcal{L}_a \mathcal{M}_a \text{tr}_b(\mathcal{P}_{a,b} \mathcal{H}_a \mathcal{S}_{a,b}^-) \mathcal{M}_a^{-1} \mathcal{T}_a \} \end{aligned}$$

where $\text{tr}_{a,b} = \text{tr}_a \text{tr}_b$. Using the fact that R_a^- is a projector, and noting (3.9), it is easily seen that we can choose the conditions

$$\text{tr}_a(\mathcal{P}_{a,b} \mathcal{L}_b \mathcal{S}_{a,b}^+) = R_b^- \tag{4.15a}$$

$$\text{tr}_b(\mathcal{P}_{a,b} \mathcal{H}_a \mathcal{S}_{a,b}^-) = 1_a. \tag{4.15b}$$

Both equations can be solved simultaneously if

$$\mathcal{L}_a = \mathcal{H}_a R_a^-$$

using the relation (4.4c). In that case, the above equality reduces to

$$\text{tr}_a(\mathcal{H}_a \mathcal{T}_a \mathcal{M}_a^{-1} R_a^- \mathcal{M}_a) = \text{tr}_a(\mathcal{H}_a R_a^- \mathcal{M}_a \mathcal{M}_a^{-1} \mathcal{T}_a).$$

Using again (3.9), we now note that

$$R_a \mathcal{M}_a \mathcal{M}_a^{-1} R_a^- = \mathcal{M}_a (R_a^-)^2 \mathcal{M}_a^{-1} = \mathcal{M}_a R_a^- \mathcal{M}_a^{-1} = R_a^- \mathcal{M}_a \mathcal{M}_a^{-1} = R_a^-$$

and similarly

$$R_a \mathcal{M}_a^{-1} R_a^- \mathcal{M}_a = \mathcal{M}_a^{-1} (R_a^-)^2 \mathcal{M}_a = \mathcal{M}_a^{-1} R_a^- \mathcal{M}_a = R_a^- \mathcal{M}_a^{-1} \mathcal{M}_a = R_a^-$$

making use of (4.11) and the fact that R_a^- is a projector, leading to

$$\mathrm{tr}_a(\mathcal{H}_a \mathcal{T}_a) = \mathrm{tr}_a(\mathcal{H}_a \mathcal{T}'_a). \quad (4.16)$$

Remark. We note that the requirement for invariance of operators of the form (4.13) in the case that the R_a^- are *not* projectors can still be met by taking

$$\begin{aligned} \mathrm{tr}_a(\mathcal{P}_{a,b} \mathcal{L}_b \mathcal{P}_{a,b}^+) &= 1_b \\ \mathrm{tr}_b(\mathcal{P}_{a,b} \mathcal{H}_a \mathcal{P}_{a,b}^-) &= R_a^- \end{aligned}$$

instead of (4.15). Both relations can be simultaneously solved by taking now

$$\mathcal{H}_a = \mathcal{L}_a \mathcal{P}_a^-.$$

However, the resulting invariants will typically factorize into lower-order invariants, because of the combination $R_a^- R_a^-$ that will occur when contracting with T_a , in combination with the unitarity condition

$$R_{12}(\lambda_1, \lambda_2) R_{21}(\lambda_2, \lambda_1) = 1$$

that is applicable to many of the well known quantum models, in particular the ones considered in section 5.

It now remains to be shown that the higher-order trace objects $\tau^{(n)}$ defined above yield a commuting family of quantum operators. This is ensured by the following statement

Proposition 2. In order for (4.13) to yield a commuting family of operators it is sufficient that the following relation holds

$$\mathcal{H}_a^{t_a} (({}^t_a \mathcal{P}_{a,b}^-)^{-1}) \mathcal{H}_b \mathcal{P}_{a,b}^+ = \mathcal{P}_{a,b}^- \mathcal{H}_b^{t_b} (({}^t_b \mathcal{P}_{a,b}^+)^{-1}) \mathcal{H}_a \quad (4.18)$$

(the left-superscripts t_a, t_b denote matrix transposition in the factors corresponding to the labels a, b in the tensor products). If, in addition, the spectral parameters λ_a and λ_b are chosen such that R_a^- and R_b^- are projectors (4.8), then it is sufficient that (4.18) holds modulo a multiplication from the left and the right by factors R_a^- and R_b^- .

Proof. In order to derive (4.18), let us give an argument similar to the one given by Sklyanin in [20] (cf also [18]). Denoting by τ_a, τ_b the invariants (4.13) evaluated at different values λ_a resp. λ_b of the spectral parameters, we have

$$\begin{aligned} \tau_a \tau_b &= \mathrm{tr}_a(\mathcal{T}_a \mathcal{H}_a) \mathrm{tr}_b(\mathcal{T}_b \mathcal{H}_b) = \mathrm{tr}_{a,b} \{ \mathcal{T}_a \mathcal{H}_a^{t_a} \mathcal{T}_b^{t_b} \mathcal{H}_b \} \\ &= \mathrm{tr}_{a,b} \{ ({}^t_b \mathcal{T}_a \mathcal{P}_{a,b}^+ \mathcal{T}_b) {}^t_a \mathcal{H}_a {}^t_a (({}^t_b \mathcal{P}_{a,b}^+)^{-1}) {}^t_b \mathcal{H}_b \} \\ &= \mathrm{tr}_{a,b} \{ (\mathcal{P}_{a,b}^+)^{-1} \mathcal{T}_b \mathcal{P}_{a,b}^- \mathcal{T}_a \mathcal{P}_{a,b}^- [{}^t_a \mathcal{H}_a {}^t_a (({}^t_b \mathcal{P}_{a,b}^+)^{-1}) {}^t_b \mathcal{H}_b] \} \\ &= \mathrm{tr}_{a,b} \{ \mathcal{T}_b \mathcal{P}_{a,b}^- \mathcal{T}_a {}^t_{ab} [({}^t_{ab} \mathcal{P}_{a,b}^+)^{-1} {}^t_a \mathcal{H}_a {}^t_a (({}^t_b \mathcal{P}_{a,b}^+)^{-1}) {}^t_b \mathcal{H}_b {}^t_{ab} \mathcal{P}_{a,b}^-] \} \end{aligned} \quad (4.19)$$

whereas on the other hand we have

$$\begin{aligned} \tau_b \tau_a &= \mathrm{tr}_b(\mathcal{T}_b \mathcal{H}_b) \mathrm{tr}_a(\mathcal{T}_a \mathcal{H}_a) \\ &= \mathrm{tr}_{a,b} \{ ({}^t_a \mathcal{T}_b \mathcal{P}_{a,b}^- \mathcal{T}_a) {}^t_b \mathcal{H}_b {}^t_b (({}^t_a \mathcal{P}_{a,b}^-)^{-1}) {}^t_a \mathcal{H}_a \} \\ &= \mathrm{tr}_{a,b} \{ \mathcal{T}_b \mathcal{P}_{a,b}^- \mathcal{T}_a {}^t_{ab} [{}^t_b \mathcal{H}_b {}^t_b (({}^t_a \mathcal{P}_{a,b}^-)^{-1}) {}^t_a \mathcal{H}_a] \} \end{aligned} \quad (4.20)$$

which leads to (4.18) identifying (4.19) and (4.20). Clearly, in the case that R_a^- and R_b^- are projectors, taking note of (4.11), it is sufficient that a weaker condition holds, namely (4.18) multiplied from the left and the right by these projectors.

It now remains to be established that the solution of (4.14), which leads to exact quantum invariants of the mapping, also obeys (4.18). This is stated by the following:

Proposition 3. If λ_a is chosen such that R_a^- is a projector, then the family of operators (4.13), where \mathcal{H}_a is a solution of (4.14), forms a commuting family of quantum operators, i.e.

$$[\tau^{(n)}(\lambda_a), \tau^{(m)}(\mu_b)] = 0 \tag{4.21}$$

for $n = \ell(a)$, $m = \ell(b)$.

Proof. The solution of (4.14) is given by†

$$\mathcal{H}_a = \text{tr}_{a'} \{ \mathcal{P}_{a', \hat{a}} \iota_a(({}^a \mathcal{P}_{a', \hat{a}}^+)^{-1}) \}. \tag{4.22}$$

This is easily verified by the observation that

$$\begin{aligned} \text{tr}_b \{ \mathcal{P}_{b, \hat{a}} \mathcal{H}_a \mathcal{P}_{b, \hat{a}}^+ \} &= \text{tr}_{a', b} \{ \mathcal{P}_{b, \hat{a}} \mathcal{P}_{a', \hat{a}} ({}^a \mathcal{P}_{a', \hat{a}}^+)^{-1} \mathcal{P}_{b, \hat{a}}^+ \} \\ &= \text{tr}_{a', b} \{ \mathcal{P}_{a', \hat{a}} \mathcal{P}_{b, \hat{a}} ({}^a \mathcal{P}_{a', \hat{a}}^+)^{-1} \mathcal{P}_{b, \hat{a}}^+ \} = \text{tr}_{a'} \{ \mathcal{P}_{a', \hat{a}} \text{tr}_b (\mathcal{P}_{b, \hat{a}} ({}^a \mathcal{P}_{a', \hat{a}}^+)^{-1} \mathcal{P}_{b, \hat{a}}^+) \} \\ &= \text{tr}_{a'} \{ \mathcal{P}_{a', \hat{a}} ({}^a \mathcal{P}_{a', \hat{a}}^+)^{-1} * \mathcal{P}_{a', \hat{a}}^+ \} = \text{tr}_{a'} (\mathcal{P}_{a', \hat{a}} 1_{a', \hat{a}}) = 1_a. \end{aligned}$$

In order to have a commuting family of invariants, it is sufficient according to proposition 2 that \mathcal{H}_a obeys (4.18) projected by factors R_a^- and R_b^- . This condition is verified by the solution (4.22) for the given choice of spectral parameters. This is checked by simply inserting \mathcal{H}_a into (4.18) and by using the relations (4.5) together with (4.2) to verify that the equation is satisfied. Details of this calculation are provided in appendix 2.

As a corollary of proposition 3, we recover under special circumstances the construction of quantum determinants associated with the algebra \mathcal{A} , namely when R_a^- is a fully antisymmetric tensor projecting out a one-dimensional subspace in the tensor product of \mathbb{C}^N . In that case, we will call the length $\ell(a)$ of the corresponding multi-index *a maximal*, and the corresponding minors \mathcal{T}_a of maximal length will be referred to as the quantum determinants of the model. We will come back to this in section 5.

Thus, we have now a complete and general construction of commutative families of exact quantum invariants of mappings (4.6) associated with the Yang–Baxter structure (4.5). We note that the requirements of having not only a commuting family of operators (for which any solution of (4.18) suffices), but in addition for these operators to be invariants under the mapping, is strong enough to uniquely determine the invariant family, i.e. it forces us to consider the specific solution of (4.18) that is given by (4.22).

Remark. An alternative way of writing the invariants $\tau^{(n)}(\lambda_a)$ is by using a tensor K_a given by

$$K_a = \mathcal{H}_a S_a^+ = \text{tr}_{a'} \{ \mathcal{P}_{a', \hat{a}} S_a^+ \iota_a(({}^a S_{a', \hat{a}}^+)^{-1}) \} \tag{4.23}$$

† In fact, for tensor objects of the form $X_{a,b}$, $Y_{a,b}$, one can introduce an associative twisted product

$$X_{a,b} * Y_{a,b} \equiv {}^b ({}^b X_{a,b} {}^b Y_{a,b}) = {}^a ({}^a Y_{a,b} {}^a X_{a,b}).$$

Then, an inverse with respect to the product $*$ is given by

$$X_{a,b}^{-1} = {}^b (({}^b X_{a,b})^{-1}) = {}^a (({}^a X_{a,b})^{-1}).$$

The solution \mathcal{H}_a of (4.22) is the contraction of the $*$ -inverse of $\mathcal{P}_{a,b}^-$.

leading to the following explicit expression for the invariants

$$\tau^{(n)}(\lambda_a) = \text{tr}_a(K_a T_a) = \text{tr}_{a,a'}\{\mathcal{P}_{a',a} S_a^{+t_a} (({}^a S_{a',a}^+)^{-1}) T_a^0 R_a^-\} \quad (n = \ell(a) = \ell(a'), \lambda_a = \lambda_{a'}) \quad (4.24)$$

and the choice of λ_a for which R_a^- is a projector. We mention, finally, that similar objects have been considered in [7] in the construction of central elements of quantum groups. However, the connection with exact invariants of quantum mappings has to our knowledge not been derived before.

5. The Gel'fand–Dikii hierarchy

We now present as special examples of the construction given in the previous sections a specific class of quantum mappings associated with the lattice Gel'fand–Dikii hierarchy [12], which is a specific class of $N \times N$ matrix lattice models whose continuum limit reduces to the usual GD hierarchy of equations. In [17] the R , S -matrix structure for the GD mappings was introduced, where we established that the full Yang–Baxter structure (2.2)–(2.6) is verified for these mappings together with the following solution of the quantum relations (2.3a, 2.3b) together with (2.4), namely

$$\begin{aligned} R_{12}^+ &= R_{12}^+(\lambda_1, \lambda_2) = R_{12}^- + \frac{\hbar}{\lambda_2} Q_{12} - \frac{\hbar}{\lambda_1} Q_{21} \\ R_{12}^- &= R_{12}^-(\lambda_1, \lambda_2) = 1 + \hbar \frac{P_{12}}{\lambda_1 - \lambda_2} \\ S_{12}^+ &= S_{12}^+(\lambda_1, \lambda_2) = S_{21}^-(\lambda_2, \lambda_1) = 1 - \frac{\hbar}{\lambda_2} Q_{12} \end{aligned} \quad (5.1)$$

in which

$$P_{12} = \sum_{i,j=1}^N E_{i,j} \otimes E_{j,i} \quad Q_{12} = \sum_{i=1}^{N-1} E_{N,i} \otimes E_{i,N} \quad (5.2)$$

the $E_{i,j}$ being the generators of GL_N in the fundamental representation, i.e. $(E_{i,j})_{kl} = \delta_{ik} \delta_{jl}$. The special case of $N=2$, leading to a quantization of the $\kappa\alpha\nu$ mappings was presented in [16]. The solution (5.1) consists of the usual $N \times N$ rational R -matrix, for which the special choice $R_{12}^-(\lambda_1, \lambda_2 = \lambda_1 \pm \hbar)$ leads to a projection matrix. In fact, choosing the spectral parameters according to

$$\lambda_a = (\lambda_1, \dots, \lambda_n) \quad \lambda_{i+1} = \lambda_i + \hbar \quad (i = 1, \dots, n-1)$$

the matrix R_a^- becomes a fully antisymmetric tensor acting on the n -fold tensor product of auxiliary vector spaces $(\mathbb{C}^N)^{\otimes n}$ (cf. eq. (4.9)).

To implement the construction of invariants for the mappings in the GD hierarchy we need the following ingredients.

(a) We consider a periodic chain of $2P$, ($P=1, 2, \dots$), sites labelled by n , and elementary matrices V_n of the form

$$V_n = \Lambda_n \left(1 + \sum_{i>j=1}^N v_{i,j}(n) E_{i,j} \right) \quad (5.3)$$

with

$$\Lambda_n = \Lambda(\lambda_n) \quad \Lambda(\lambda) = \lambda E_{N,1} + \sum_{i=1}^{N-1} E_{i,i+1} \quad \lambda_{2n} = \lambda \quad \lambda_{2n+1} = \lambda + \omega \quad (5.4)$$

where

$$v_{i,j} = 0 \quad (i \neq N, j \neq 1, i \neq j+1) \quad v_{i+1,i}(n) = p_{n+1} \quad (i = 2, \dots, N-2)$$

the p_n being constant parameters such that $p_{2n} = p_{2n+2}$, and where the only operator-valued entries are $v_{N,j}$ and $v_{i+1,1}$, ($i, j = 1, \dots, N-1$), where $\omega = (-p_{2n})^N - (-p_{2n+1})^N$, at each site of the chain. The matrices V_n depends on a different spectral parameter λ or $\lambda + \omega$ depending on the even or odd site of the chain. We impose now for the matrices V_n the commutation relations

$$V_{n+1,1} S_{n,12}^+ V_{n,2} = V_{n,2} V_{n+1,1} \quad (5.5a)$$

$$R_{n,12}^+ V_{n,1} V_{n,2} = V_{n,2} V_{n,1} R_{n,12}^- \quad (5.5b)$$

$$V_{n,1} V_{m,2} = V_{m,2} V_{n,1} \quad |n - m| \geq 2 \quad (5.5c)$$

in which S_n^\pm and R_n^\pm depend on the local spectral parameters $\lambda_{n,1}$ resp. $\lambda_{n,2}$ associated with the site n , i.e. as in (5.1) with λ replaced by λ_n . The entries of the matrices V_n do not depend on the spectral parameter λ , and are Hermitean operators $v_{i,j}$ obeying the following Heisenberg type of commutation relations, ($\hbar = i\hbar$)

$$[v_{i,j}(n), v_{k,l}(m)] = \hbar(\delta_{n,m+1} \delta_{k,j+1} \delta_{i,N} \delta_{l,1} - \delta_{m,n+1} \delta_{i,l+1} \delta_{k,N} \delta_{j,1}) \quad (5.6)$$

in agreement with (5.5a) and (5.5b).

(b) With the identification

$$L_n(\lambda) = V_{2n}(\lambda) V_{2n-1}(\lambda + \omega) \quad (5.7)$$

we have a solution of the quantum relations (2.2), and the quantum M -matrix is given by

$$M_n = \Lambda_{2n} \left(1 + \sum_{i=2}^{N-1} v'_{i,1}(2n-2) E_{i,1} + p_{2n} \sum_{j=2}^{N-2} E_{j+1,j} + \sum_{j=1}^{N-1} v_{N,j}(2n-1) E_{N,j} + w(n) E_{N,1} + (p_{2n} - p_{2n+1}) E_{2,1} \right). \quad (5.8)$$

The corner entry $w(n)$ is determined by the Zakharov–Shabat equations (2.1). For the matrix M_n we have the commutation relations

$$M_{n+1,1} S_{12}^+ V_{2n,2} = V_{2n,2} M_{n+1,1} \quad (5.9a)$$

$$V'_{2n-1,1} S_{12}^+ M_{n,2} = M_{n,2} V'_{2n-1,1} \quad (5.9b)$$

$$[M_{n,1}, V_{2n-k,2}] = [M_{n+1,1}, V'_{2n-k,2}] = 0 \quad (k \neq 2, 1, 0, -1). \quad (5.9c)$$

(c) One can work out the zs equation (2.1) to obtain explicit expressions for the entries of the M -matrix in terms of the entries of the V_n . The result is rather complicated

$$v'_{2,1}(2n-1) = v_{2,1}(2n) \quad (5.10a)$$

$$v'_{i,1}(2n-1) + p_{2n+1} v'_{i-1,1}(2n-1) = v_{i,1}(2n) + p_{2n} v_{i-1,1}(2n), \quad (i = 3, \dots, N-1) \quad (5.10b)$$

$$v'_{N,N-1}(2n-1) = v_{N,N-1}(2n) + p_{2n} - p_{2n+1} \quad (5.10c)$$

$$v'_{N,j}(2n-1) = v_{N,j}(2n) \quad (j=2, \dots, N-2) \quad (5.10d)$$

$$v'_{N,1}(2n-1) + p_{2n+1}v'_{N-1,1}(2n-1) = v_{N,1}(2n) + p_{2n}v_{N-1,1}(2n) \quad (5.10e)$$

$$v'_{3,1}(2n-2) - v_{3,1}(2n-1) = v_{2,1}(2n)[p_{2n+1} - p_{2n} + v_{2,1}(2n-1) - v'_{2,1}(2n-2)] \\ + (w(n) - v'_{3,1}(2n-2))\delta_{N,3} \quad (5.10f)$$

$$v'_{N,N-2}(2n) = v_{N,N-2}(2n+1) - p_{2n}v'_{N,N-1}(2n) + p_{2n+1}v_{N,N-1}(2n+1) \\ + (v'_{N,N-1}(2n) - v_{N,N-1}(2n+1))(p_{2n+1} - v_{N,N-1}(2n)) \\ + (w(n+1) - v_{3,1}(2n+1))\delta_{N,3} \quad (5.10g)$$

$$v'_{i,1}(2n-2) = v_{i,1}(2n-1) - p_{2n+1}[v'_{i-1,1}(2n-2) - v_{i-1,1}(2n-1)] \\ + v_{i-1,1}(2n)[p_{2n+1} - p_{2n} + v_{2,1}(2n-1) - v'_{2,1}(2n-2)] \\ - (w(n) - v'_{N,1}(2n-2))\delta_{N,i} \quad (i=4, \dots, N) \quad (5.10h)$$

$$v'_{N,j}(2n) = v_{N,j}(2n+1) - p_{2n}v'_{N,j+1}(2n) + p_{2n+1}v_{N,j+1}(2n+1) \\ - (v'_{N,N-1}(2n) - v_{N,N-1}(2n+1))v_{N,j+1}(2n) + (w(n+1) \\ - v_{N,1}(2n+1))\delta_{j,1} \quad (j=1, \dots, N-3) \quad (5.10i)$$

$$\lambda_{2n-1} - \lambda_{2n} = \sum_{k=2}^{N-2} v_{N,k}(2n+1)v_{k+1,1}(2n) - \sum_{k=2}^{N-2} v'_{N,k}(2n)v'_{k+1,1}(2n-1) \\ + (w(n+1) - v'_{N,1}(2n))(v_{2,1}(2n) + p_{2n}) \\ - (v'_{N,N-1}(2n) - v_{N,N-1}(2n+1))(v_{N,1}(2n) + p_{2n}v_{N,2}(2n)) \\ - v'_{N,N-1}(2n)(p_{2n}v_{N-1,1}(2n-2) + p_{2n+1}v'_{N-1,1}(2n-1)) \\ + p_{2n}(p_{2n+1}v_{N,2}(2n+1) - p_{2n}v'_{N,2}(2n)) \quad (5.10j)$$

$$\lambda_{2n} - \lambda_{2n-1} = \sum_{k=2}^{N-2} v_{N,k}(2n)v_{k+1,1}(2n-1) - \sum_{k=2}^{N-2} v'_{N,k}(2n-1)v'_{k+1,1}(2n-2) \\ - (v_{N,N-1}(2n) + p_{2n})(w(n) - v_{N,1}(2n-1)) \\ - (v_{N,1}(2n) + p_{2n}v_{N,N-1}(2n))(v'_{2,1}(2n-2) - v_{2,1}(2n-1) + p_{2n} - p_{2n+1}) \\ - p_{2n}p_{2n+1}(v'_{N-1,1}(2n-2) - v_{N-1,1}(2n-1)) \quad (5.10k)$$

$$v_{N,N-1}(n) = p_{n+2} - v_{2,1}(n-2). \quad (5.10l)$$

The mapping as expressed by (5.10) in terms of the operators $v_{i,j}$ is not very enlightening, and there is a more convenient set of variables, $X_n^{(\alpha)}$, ($\alpha=1, \dots, N-1$, $n=1, \dots, 2P$) introduced in [17] (cf also [12]) in terms of which the mapping seems more natural. However, the variables $v_{i,j}$ are the natural ones in connection with the quantum mapping algebra (2.2)–(2.5).

From the explicit form of (5.10) it can be shown by explicit calculation that the mapping is symplectic, i.e. it preserves the basic commutation relations. Therefore, the relations (5.5) are preserved under the mapping. A more fundamental reason for the symplecticity is the existence of an action for the entire family of GD mappings [28] (cf also [14, 27]) for the KdV case. A next step is to work out the commutation relations between the operators V_n and M_n , using the explicit expressions that one obtains for

the entries of M_n in terms of V_n and V'_n . This is a fairly elaborate, but straightforward calculation, and we omit the details. (Some details of the calculation are given in appendix 3).

(d) Having established the full Yang–Baxter structure for the mapping (5.10), we can use now the formalism of the previous section, we can calculate the explicit K -matrices that lead to commuting families of exact quantum invariants. The monodromy matrix $T(\lambda)$ is constructed from the matrices V_n by

$$T(\lambda) \equiv \prod_{n=1}^{\leftarrow 2P} V_n(\lambda). \tag{5.11}$$

The invariants for the GD hierarchy are given by (4.24), namely

$$\tau^{(n)}(\lambda) = \text{tr}_{1, \dots, n}(K_{(1, \dots, n)} T_{(1, \dots, n)}) = \text{tr}_{1, \dots, n}(K_{(1, \dots, n)} T_{(n, \dots, 1)}^0 P_{(1, \dots, n)}^-) \tag{5.12}$$

($n = 1, \dots, N$)

where

$$T_{(n, \dots, 1)}^0 = T_n S_{n, n-1}^+ T_{n-1} S_{n, n-2}^+ S_{n-1, n-2}^+ T_{n-2} \dots T_2 S_{n, 1}^+ S_{n-1, 1}^+ \dots S_{2, 1}^+ T_1 \tag{5.13}$$

$\lambda_i = \lambda + (i - 1)h \quad (i = 1, \dots, n)$

and where $P_{(1, \dots, n)}^-$ denotes the antisymmetrizer as in (4.10). The K -matrix which is obtained from (4.24) turns out to factorize due to the nilpotency of the S -matrix of (5.1). In fact, we find

$$K_{(1, \dots, n)} = K_1 (S_{2, 1}^+)^{-1} K_2 (S_{3, 2}^+)^{-1} (S_{3, 2}^+)^{-1} K_3 \dots K_{n-1} (S_{n, 1}^+)^{-1} \dots (S_{n, n-1}^+)^{-1} K_n \tag{5.14}$$

$$K_i(\lambda_i) = 1 + \frac{h}{\lambda_i} Q_{i, i} = 1 + (N - 1) \frac{h}{\lambda_i} (E_{N, N})_i \quad \lambda_i = \lambda + (i - 1)h$$

in which $Q_{i, i}$ denotes the contraction of the tensor Q of (5.2). The actual invariants are now obtained by expanding in powers of the spectral parameter λ . Due to the form of the matrices V_n , (5.3), the monodromy matrix (5.11) has coefficients which are simply polynomial in λ . It is a matter of counting to verify that by considering all invariants thus obtained from (5.12) together with (5.14), for $n = 1, \dots, N$, will yield a sufficiently large family of commuting invariants of the quantum mapping in terms of the reduced variables $X_n^{(\alpha)}$ introduced in [12, 17]. We shall not occupy ourselves here with this problem, as we envisage to deal with that issue at a later date when we plan to investigate the representation theory for the mapping algebra [28]. The Casimirs for the mapping algebra \mathcal{A} are given by the coefficients of the deformed quantum determinant [35], which is obtained from (5.12) for the top value, $n = N$.

Remark. The role played by the ‘pivot’ matrix $\Lambda(\lambda)$ in the above construction of the GD-hierarchy can be made clear by the following. Due to the identities

$$R_{12}^+ = \Lambda_1 \Lambda_2 R_{12} \Lambda_1^{-1} \Lambda_2^{-1} \quad S_{12}^+ = \Lambda_2 S_{12} \Lambda_2^{-1}$$

in which

$$R_{12} \equiv R_{12}^- \quad S_{12} \equiv \Lambda_2^{-1} S_{12}^+ \Lambda_2$$

it is evident that in this case it is not strictly necessary to introduce two different R -matrices R^\pm . In fact, taking $R_{12} \equiv R_{12}^-$ we can go over to a ‘symmetric’ version of the

quantum relations (2.2a), namely

$$R_{12}\mathcal{L}_{n,1}\mathcal{L}_{n,2} = \mathcal{L}_{n,2}\mathcal{L}_{n,1}R_{12} \quad \mathcal{L}_{n+1,1}S_{12}\mathcal{L}_{n,2} = \mathcal{L}_{n,2}\mathcal{L}_{n+1,1} \quad (5.15)$$

for $\mathcal{L} \equiv \Lambda^{-1}L$. However, working with this symmetrized version of the basic equations has the disadvantage that the relations (2.4) and the definition of the monodromy matrix become less natural. The relation (2.4) reduces then to

$$R_{12}\Lambda_1 S_{21}\Lambda_2 = \Lambda_2 S_{12}\Lambda_1 R_{12}.$$

In the case of the GD-hierarchy the symmetric R - and S -matrices are given by

$$R_{12} = 1 + h \frac{P_{12}}{\lambda_1 - \lambda_2} \quad S_{12} = 1 - h \sum_{i=1}^{N-1} E_{N,i} \otimes E_{i+1,1}. \quad (5.16)$$

6. Conclusions

We have given a general construction of exact quantum invariants associated with the quantum mapping algebra given by eqs. (2.2)–(2.5). The relevant algebra for the monodromy matrix is given by the equations (2.9) together with (2.14) and (2.13). The construction of commuting families of quantum operators follows basically the same philosophy as the construction in [20]. As a consequence, weak integrability of the mappings (in the terminology of [17]) is established by developing the fusion algebra associated with the mapping algebra, which ensures that commuting families of (not necessarily invariant) operators are mapped again to commuting families. However, we have shown under what conditions the statement of strong integrability of mappings can also be made for the algebra \mathcal{A} of the quantum mappings. It turns out that there is actually a unique family of commuting operators made up of exact quantum *invariants* of the mapping. Implementing the fusion procedure we have established that this invariance can be pushed to the level of higher-order quantum minors and determinants, associated with e.g. higher-spin models or (as we have demonstrated by the example of the GD hierarchy) $N \times N$ matrix quantum models. These results now open the way to the exact ‘diagonalization’ of the quantum discrete-time evolution, e.g. via the algebraic Bethe’s Ansatz [2] applied to the quantum mapping algebra.

We have not embarked in this paper on issues of representation theory, in terms of which the results presented here can no doubt be generalized. Another issue that we have not addressed here is the construction of the generating quantum operator of the discrete-time flow. A construction of such operators has been given recently in [40] in the context of the quantum Volterra model. However, for the mappings of the GD hierarchy, a direct connection between the unitary operator of the quantum canonical transformation, as was derived in [27], and the monodromy matrix of the system, has still to be established. A direct construction of these operators on the basis of quantum actions for the mappings is under investigation [28].

We finish by mentioning in this context also the recent interest in q -difference analogues of the Knizhnik–Zamolodchikov equations, that have appeared recently in connection with the representation theory of affine quantum groups [38, 39]. This is yet another manifestation of the inherent discrete nature of quantum groups and

related objects. It would be of interest to bring all these aspects together in one global difference approach to the underlying structures.

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Appendix 1

In this appendix we derive the relations (3.4), (3.5) and (3.6). The proofs of (3.3) are obtained by direct iteration of the Yang–Baxter equations (2.3).

(i) Equations (3.4) can be proven by induction. They hold trivially for the case $\ell(a) = 1$. For the induction procedure it suffices to show how the equations can be build up if we merge multi-indices, thus showing that from smaller multi-indices we can construct the same statements for multi-indices made up of the smaller pieces. (This will be the philosophy for all the proofs.) Thus, assuming that (3.4c) hold for multi-indices a, b, \dots , we show that they also hold for merged indices such as (ab) etc. For instance, if we want to demonstrate (3.4c), we perform the following sequence of steps

$$\begin{aligned}
 R_{(ab)c}^{\pm} S_{c,(ab)}^{\pm} &= R_{\hat{a},b}^{\pm} R_a^{\pm} R_b^{\pm} S_{c,a}^{\pm} S_{c,b}^{\pm} \\
 &= R_{\hat{a},b}^{\pm} S_{c,a}^{\pm} S_{c,b}^{\pm} R_a^{\pm} R_b^{\pm} \\
 &= S_{c,b}^{\pm} S_{c,a}^{\pm} R_{\hat{a},b}^{\pm} R_a^{\pm} R_b^{\pm} \\
 &= S_{c,(ab)}^{\pm} \widehat{R}_{(ab)}^{\pm} \quad (\widehat{ab}) = (b\hat{a})
 \end{aligned}
 \tag{A1.1}$$

and similarly for the other relations.

(ii) Equation (3.5) can again be simply proven by induction. Equation (3.5) clearly holds for $\ell(a) = 1$, in which case the objects R_a^{\pm} and S_a^{\pm} are simply the unit matrices acting on one single vector space. Now, assuming that (3.5) holds for a fixed value of $\ell(a)$, then we can iterate as follows

$$\begin{aligned}
 R_{\hat{a},(bc)}^{-} S_{(bca)}^{+} &= R_{\hat{a},b}^{-} R_{\hat{a},c}^{-} S_{(ca)}^{+} S_{\hat{b},(ca)}^{+} S_b^{+} \\
 &= R_{\hat{a},b}^{-} S_{(ac)}^{+} R_{\hat{a},c}^{-} S_{\hat{b},a}^{+} S_{\hat{b},c}^{+} S_b^{+} \\
 &= R_{\hat{a},b}^{-} S_c^{+} S_{\hat{a},c}^{+} S_a^{+} S_{\hat{b},c}^{+} S_{\hat{b},a}^{+} S_b^{+} R_{\hat{a},c}^{-} \\
 &= S_c^{+} S_{\hat{b},c}^{-} S_{\hat{b},a}^{-} R_{\hat{a},b}^{-} S_a^{+} S_{\hat{b},a}^{+} S_b^{+} R_{\hat{a},c}^{-} \\
 &= S_c^{+} S_{\hat{b},c}^{+} S_{\hat{a},c}^{+} S_{(ab)}^{+} R_{\hat{a},b}^{+} R_{\hat{a},c}^{+} = S_{(abc)}^{+} R_{\hat{a},(bc)}^{+}
 \end{aligned}
 \tag{A1.2}$$

where in the first and last step we have used the decomposition of $S_{(abc)}^{\pm}$ according to (3.1), as well as the induction assumption, and in the second and third step the fused Yang–Baxter relations (3.3b). A similar line of steps leads to

$$R_{(\hat{ab}),c}^{-} S_{(cab)}^{+} = S_{(abc)}^{+} R_{(\hat{ab}),c}^{+}
 \tag{A1.3}$$

thus we can break down the multi-indices in parts repeating the relations (A1.1) and (A1.2) in successive steps.

(iii) In order to prove (3.6), we break down the multi-index relation as follows

$$\begin{aligned}
 S_{(ab)}^+ R_{(ab)}^+ &= S_b^+ S_{a,b}^+ S_a^+ R_b^+ R_{a,b}^+ R_a^+ \\
 &= S_b^+ R_b^+ S_{a,b}^+ S_a^+ R_{a,b}^+ R_a^+ \\
 &= R_b^- S_b^+ S_{a,b}^+ S_a^+ R_{a,b}^+ R_a^+ \\
 &= R_b^- S_{(ab)}^+ R_{a,b}^+ R_a^+ \\
 &= R_b^- R_{a,b}^- S_{(ba)}^+ R_a^+ \\
 &= R_b^- R_{a,b}^- S_a^+ S_{b,a}^+ S_b^+ R_a^+ \\
 &= R_b^- R_{a,b}^- R_a^- S_a^+ S_{b,a}^+ S_b^+ \\
 &= R_{(ab)}^- S_{(ba)}^+ = R_{(ab)}^- S_{(ab)}^+ \tag{A1.4}
 \end{aligned}$$

where use has been made of the decomposition of $S_{(ab)}^\pm$ in the first, fourth and last step, the relation (3.4c) in the second step and penultimate step and the induction assumption in the other step.

Appendix 2

In this appendix, we give details of the verification that (4.22) obeys (4.18) when multiplied by the projectors R_a^- and R_b^- at both sides. In fact, inserting \mathcal{K}_a into the right-hand side of (4.18) we perform the following sequence of manipulations

$$\begin{aligned}
 \mathcal{K}_b {}^{t_b} \mathcal{S}_{(a,b)}^+ {}^{-1} \mathcal{K}_a (\mathcal{R}_{a,b}^+)^{-1} &= \text{tr}_{a',b'} \{ \mathcal{P}_{b',b} {}^{t_b} (({}^{t_b} \mathcal{S}_{b',b}^+)^{-1}) {}^{t_a} (({}^{t_a} \mathcal{S}_{a,b}^+)^{-1}) \mathcal{P}_{a',a} {}^{t_a} (({}^{t_a} \mathcal{S}_{a',a}^+)^{-1}) (\mathcal{R}_{a,b}^+)^{-1} \} \\
 &= \text{tr}_{a',b'} \{ \mathcal{P}_{a',a} \mathcal{P}_{b',b} {}^{t_a} {}^{t_b} [({}^{t_b} \mathcal{S}_{b',b}^+)^{-1} ({}^{t_a} \mathcal{S}_{a',a}^+)^{-1} ({}^{t_a} \mathcal{S}_{a',a}^+)^{-1} (\mathcal{R}_{a,b}^+)^{-1}] \} \\
 &= \text{tr}_{a',b'} \{ \mathcal{P}_{a',a} \mathcal{P}_{b',b} {}^{t_a} {}^{t_b} [({}^{t_b} \mathcal{S}_{b',b}^+)^{-1} (\mathcal{R}_{a,b}^+)^{-1} ({}^{t_a} \mathcal{S}_{a',a}^+)^{-1} ({}^{t_a} \mathcal{S}_{a',a}^+)^{-1}] \} \\
 &= \text{tr}_{a',b'} \{ {}^{t_a} \mathcal{P}_{a',a} {}^{t_b} \mathcal{P}_{b',b} {}^{t_a} {}^{t_b} \mathcal{S}_{b',a}^+ [({}^{t_b} \mathcal{S}_{b',a}^+)^{-1} ({}^{t_b} \mathcal{S}_{b',b}^+)^{-1} (\mathcal{R}_{a,b}^+)^{-1}] ({}^{t_a} \mathcal{S}_{a',a}^+)^{-1} ({}^{t_a} \mathcal{S}_{a',a}^+)^{-1} \} \\
 &= \text{tr}_{a',b'} \{ {}^{t_a} \mathcal{P}_{a',a} {}^{t_b} \mathcal{P}_{b',b} {}^{t_a} {}^{t_b} \mathcal{S}_{b',a}^+ (\mathcal{R}_{a,b}^+)^{-1} ({}^{t_b} \mathcal{S}_{b',b}^+)^{-1} ({}^{t_b} \mathcal{S}_{b',a}^+)^{-1} ({}^{t_a} \mathcal{S}_{a',a}^+)^{-1} ({}^{t_a} \mathcal{S}_{a',a}^+)^{-1} \}. \tag{A2.1}
 \end{aligned}$$

Multiplying at this point (A2.1) at the right by $R_a^- R_b^-$, we can use (4.4c) to move these through the matrices \mathcal{S} . Next we multiply (A2.1) at the left by the same factors and use the relations

$${}^{t_b} \mathcal{P}_{a,a} X_a = {}^{t_a} \mathcal{P}_{a,a} {}^{t_a} X_a \qquad X_a {}^{t_a} \mathcal{P}_{a,a} = {}^{t_a} X_a {}^{t_a} \mathcal{P}_{a,a} \tag{A2.2}$$

which are a consequence of (4.12), to transpose R_a^- and R_b^- into ${}^{t_a} R_a^-$ and ${}^{t_b} R_b^-$, and bring them under the trace. Then applying the cyclicity of the double trace we move these factors to the right and, by using again (4.4c), through the matrices \mathcal{S} . In this way we find

$$\begin{aligned}
 R_a^- R_b^- \mathcal{K}_b {}^{t_b} \mathcal{S}_{(a,b)}^+ {}^{-1} \mathcal{K}_a (\mathcal{R}_{a,b}^+)^{-1} R_a^- R_b^- &= \text{tr}_{a',b'} \{ {}^{t_a} \mathcal{P}_{a',a} {}^{t_b} \mathcal{P}_{b',b} {}^{t_a} {}^{t_b} \mathcal{S}_{b',a}^+ (\mathcal{R}_{a,b}^+)^{-1} R_a^- R_b^- {}^{t_a} R_a^- {}^{t_b} R_b^- ({}^{t_b} \mathcal{S}_{b',b}^+)^{-1} \\
 &\quad \times ({}^{t_b} \mathcal{S}_{b',a}^+)^{-1} ({}^{t_a} \mathcal{S}_{a',a}^+)^{-1} ({}^{t_a} \mathcal{S}_{a',a}^+)^{-1} \}. \tag{A2.3}
 \end{aligned}$$

Then, we can use again (A2.2), and subsequently (4.4c), the Yang–Baxter equations and (4.2) together with the property that R_a^- and R_b^- are projectors, to transmute the combination

$$\begin{aligned}
 & {}^a\mathcal{P}_{a',a} {}^b\mathcal{P}_{b',b} {}^b\mathcal{S}_{b',a}^+ (\mathcal{R}_{a,b}^+)^{-1} R_a^- R_b^- {}^a\mathcal{R}_{a',a}^- {}^b\mathcal{R}_{b',b}^- \\
 &= {}^a\mathcal{P}_{a',a} {}^b\mathcal{P}_{b',b} R_a^- R_b^- \mathcal{S}_{b',a}^+ (\mathcal{R}_{a,b}^+)^{-1} R_a^- R_b^- \\
 &= {}^a\mathcal{P}_{a',a} {}^b\mathcal{P}_{b',b} R_a^- R_b^- \mathcal{S}_{b',a}^+ R_a^- R_b^- (\mathcal{R}_{a,b}^+)^{-1} R_a^- R_b^- \\
 &= {}^a\mathcal{P}_{a',a} {}^b\mathcal{P}_{b',b} \widehat{R_a^- R_b^-} \mathcal{S}_{b',a}^+ (\mathcal{R}_{a,b}^+)^{-1} R_a^- R_b^- \\
 &= {}^a\mathcal{P}_{a',a} {}^b\mathcal{P}_{b',b} R_a^- R_b^- (\mathcal{R}_{a,b}^-)^{-1} \mathcal{S}_{a,b}^+ R_a^- R_b^- \\
 &= {}^a\mathcal{P}_{a',a} {}^b\mathcal{P}_{b',b} R_a^- R_b^- (\mathcal{R}_{a,b}^-)^{-1} R_a^- R_b^- \mathcal{S}_{a,b}^+ R_a^- R_b^- \\
 &= {}^a\mathcal{P}_{a',a} {}^b\mathcal{P}_{b',b} {}^a\mathcal{S}_{a',a}^+ {}^b\mathcal{S}_{b',b}^+ (\mathcal{R}_{a',b}^-)^{-1} R_a^- R_b^- {}^a\mathcal{R}_{a',a}^- {}^b\mathcal{R}_{b',b}^-. \tag{A2.4}
 \end{aligned}$$

Using then again (4.4c) and cyclicity of the trace to move the factors R_a^- , R_b^- , ${}^a\mathcal{R}_{a',a}^-$ and ${}^b\mathcal{R}_{b',b}^-$ to the right and out of the trace again, we obtain (A2.3)

$$\begin{aligned}
 & R_a^- R_b^- \text{tr}_{a',b'} \{ {}^a\mathcal{P}_{a',a} {}^b\mathcal{P}_{b',b} {}^a\mathcal{S}_{a',a}^+ {}^b\mathcal{S}_{b',b}^+ (\mathcal{R}_{a,b}^-)^{-1} ({}^b\mathcal{S}_{b',b}^+)^{-1} ({}^a\mathcal{S}_{a',a}^+)^{-1} ({}^b\mathcal{S}_{b',a}^+)^{-1} ({}^a\mathcal{S}_{a',a}^+)^{-1} \} R_a^- R_b^- \\
 &= R_a^- R_b^- \text{tr}_{a',b'} \{ {}^a\mathcal{P}_{a',a} {}^b\mathcal{P}_{b',b} (\mathcal{R}_{a,b}^-)^{-1} ({}^a\mathcal{S}_{a',a}^+)^{-1} ({}^b\mathcal{S}_{b',a}^+)^{-1} ({}^a\mathcal{S}_{a',a}^+)^{-1} \} R_a^- R_b^- \\
 &= R_a^- R_b^- \text{tr}_{a',b'} \{ \mathcal{P}_{a',a} {}^b\mathcal{P}_{b',b} (\mathcal{R}_{a,b}^-)^{-1} {}^a (({}^a\mathcal{S}_{a',a}^+)^{-1}) {}^b (({}^b\mathcal{S}_{b',a}^+)^{-1}) {}^b (({}^b\mathcal{S}_{b',b}^+)^{-1}) \} R_a^- R_b^- \\
 &= R_a^- R_b^- (\mathcal{R}_{a,b}^-)^{-1} \text{tr}_{a',b'} \{ \mathcal{P}_{a',a} {}^a (({}^a\mathcal{S}_{a',a}^+)^{-1}) {}^b (({}^b\mathcal{S}_{b',a}^+)^{-1}) \} \text{tr}_{b',b} \{ \mathcal{P}_{b',b} {}^b (({}^b\mathcal{S}_{b',b}^+)^{-1}) \} R_a^- R_b^- \\
 &= R_a^- R_b^- (\mathcal{R}_{a,b}^-)^{-1} \mathcal{H}_a {}^a (({}^a\mathcal{S}_{a,b}^-)^{-1}) \mathcal{H}_b R_a^- R_b^- \tag{A2.5}
 \end{aligned}$$

leading to (4.18).

Appendix 3

In this appendix we discuss the commutation relations for the matrices M_n as given by (5.9) and (2.6). From the explicit form of the mapping (5.10) one observes the following:

$$w(n) - v_{N,1}(2n-1) = \mathcal{A}(v'_{2,1}(2n-2) - v_{2,1}(2n-1), \{v_{j,1}(2n)\}_{j=2,\dots,N-1}) \tag{A3.1a}$$

$$w(n+1) - v'_{N,1}(2n)$$

$$= \mathcal{B}(v'_{N,N-1}(2n) - v_{N,N-1}(2n+1), \{v_{N,j}(2n), v_{N,j}(2n+1)\}_{j=2,\dots,N-1}) \tag{A3.1b}$$

in which \mathcal{A} and \mathcal{B} are some functions that can be explicitly inferred by solving $v'_{i,1}(2n-2) - v_{i,1}(2n-1)$ and $w(n) - v_{N,1}(2n-1)$ iteratively from (5.10f) and (5.10h) and solving $v'_{N,j}(2n) - v_{N,j+1}(2n+1)$ and $v'_{N,1}(2n) - w(n+1)$ from (5.10i). From (5.10j) and (5.10k), using the relations obtained for $v'_{i,1}(2n-2) - v_{i,1}(2n-1)$ and $v'_{N,j}(2n) - v_{N,j+1}(2n+1)$, we find

$$w(n) - v_{N,1}(2n-1) = \mathcal{C}(v'_{2,1}(2n-2) - v_{2,1}(2n-1)$$

$$V_{2n}, \{v_{j,1}(2n-1)\}_{j=3,\dots,N-1}) \tag{A3.2a}$$

$$w(n+1) - v'_{N,1}(2n) = \mathcal{D}(v'_{N,N-1}(2n) - v_{N,N-1}(2n+1)$$

$$V_{2n}, \{v_{N,j}(2n), v_{N,j}(2n+1)\}_{j=2,\dots,N-1}) \tag{A3.2b}$$

with explicit forms for \mathcal{C} and \mathcal{D} , which we do not specify. From (A3.1b) and (A3.2b) one can solve

$$v'_{N,N-1}(2n) - v_{N,N-1}(2n+1) = \mathcal{C}(V_{2n}, \{v_{N,j}(2n+1)\}_{j=2,\dots,N-1}) \quad (\text{A3.3})$$

and from (A3.1a) and (A3.2a) it follows that

$$v'_{2,1}(2n-2) - v_{2,1}(2n-1) = \mathcal{F}(V_{2n}, \{v_{j,1}(2n-1)\}_{j=3,\dots,N-1}) \quad (\text{A3.4})$$

\mathcal{C} and \mathcal{F} denoting some explicit expressions of the given arguments. Using (5.10i) in combination with (A3.3) and (A3.4) it follows that

$$v'_{2,1}(2n-2) - v_{2,1}(2n-1) = v'_{N,N-1}(2n) - v_{N,N-1}(2n+1) = \mathcal{G}(V_{2n}) \quad (\text{A3.5})$$

depending only on a given function \mathcal{G} of V_{2n} . Inserting (A3.5) into (5.10f) and (5.10h) we immediately obtain the relations

$$M_n - V_{2n-1} = \mathcal{H}(V_{2n}) \quad (\text{A3.6})$$

also depending only on V_{2n} , implying that M_n commutes with V_{2n-j} for $j \neq 2, 1, 0, -1$ as in (5.9c). Equation (5.9a) follows from the commutation relation $[M_{n+1} - V_{2n+1}^{\otimes}, V_{2n}] = 0$, as in (5.5a). Furthermore, from (A3.5), (5.10g) and (5.10i) we have a relation of the form

$$V'_{2n} - M_{n+1} = \mathcal{I}(V_{2n}, \{v_{N,j}(2n+1)\}_{j=2,\dots,N-1}). \quad (\text{A3.7})$$

From the fact that all elements of V'_{2n+1} can be expressed in terms of V_{2n+2} (cf eqs (5.10a)–(5.10e)), we have that $[M_{n+1} - V'_{2n}, V'_{2n+1}] = 0$, and (5.9b) follows taking into account that the mapping is symplectic.

Considering that from (A3.6) and (A3.7) we have

$$V'_{2n} - V_{2n+1} = \mathcal{H}(V_{2n+2}) + \mathcal{I}(V_{2n}, \{v_{N,j}(2n+1)\}_{j=2,\dots,N-1}) \quad (\text{A3.8})$$

it is seen that from all the V'_{2n-2k} , only V'_{2n} and V'_{2n-2} can give rise to terms involving V_{2n-1} , V_{2n} and V_{2n+1} that have non-vanishing commutation relations with $V'_{2n} - M_{n+1}$. Hence $[M_{n+1} - V'_{2n-k}, V'_{2n-k}] = 0$, for $k \neq 2, 1, 0, -1$, as in (5.9c). From (A3.6) we have

$$[M_n, v_{N,j}(2n+1)] = 0 \quad (j = 2, \dots, N-1). \quad (\text{A3.9})$$

Considering the relation

$$v'_{i,1}(2n-2) - v_{i,1}(2n-1) = \mathcal{J}(V_{2n}) \quad (\text{A3.10})$$

which follows from (A3.5) in combination with (5.10f) and (5.10h), it follows that $v'_{i,1}(2n-1)$ can only have non-vanishing commutation relations with V_{2n+1} , V_{2n} and $v_{i,1}(2n-1)$. Taking into account (A3.7) and the symplecticity of the mapping, we find

$$[M_n, v'_{i,1}(2n-2)] = 0 \quad (j = 2, \dots, N-1). \quad (\text{A3.11})$$

Equation (A3.9) and (A3.11) can be combined to yield that $[M_n^{\otimes}, M_n] = 0$, from which (2.6a) can be derived in analogy with (5.5b) for the matrix V_n . Finally, from the updated ('primed') version of (A3.6) and the commutation relation $[V'_{2n}, M_n] = 0$, we obtain

$$[M'_n - V'_{2n-1}, M_n] = 0 \quad (\text{A3.12})$$

leading with (5.9b) to (2.6b). This proves all the commutation relations between the matrices V_n and M_n as given in section 5.

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